Types of Matrices and Diagonalization of Matrices

Symmetric matrix: A square matrix is called symmetric matrix if $A = A^T$

i.e.$a_{ij} = a_{ji}$

e.g.$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 7 \end{bmatrix}$

Skew-Symmetric matrix: A square matrix is called symmetric matrix if $A = -A^T$

i.e. $a_{ij} = -a_{ji}$. The diagonal elements of a skew-symmetric matrix are zero because $a_{ii} = -a_{ii}$ if and only if $a_{ii} = 0$

e.g. $\begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & 6 \\ -3 & -6 & 0 \end{bmatrix}$

Orthogonal matrix: A square matrix A is said to be orthogonal if

$$AA^T = A^T A = I.$$

$$\text{e.g. } A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

Unitary matrix: A square matrix A is said to be Unitary if

$$A^\theta A = AA^\theta = I$$

where $A^\theta = \left(\overline{A}\right)^T$

e.g. $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$

Note: Every orthogonal matrix is unitary.

Hermitian matrix: A square matrix A is said to be Hermitian matrix if

$A^\theta = A$ i.e. $a_{ij} = \overline{a_{ji}}$

Diagonal elements of a Hermitian matrix are real numbers.

$$\text{e.g. } A = \begin{bmatrix} 1 & 2+3i & 5-6i \\ 2-3i & 2 & 9-6i \\ 5+6i & 9+6i & -11 \end{bmatrix}$$

Skew Hermitian matrix: A square matrix A is said to be skew Hermitian matrix if

$A^{\theta} = -A$ i.e. $a_{ij} = -\overline{a_{ji}}$

Diagonal elements of a skew Hermitian matrix are either zero or purely imaginary numbers.

$$\text{e.g. } A = \begin{bmatrix} 1 & 2+3i & -5-6i \\ -2+3i & 2 & -9+6i \\ 5-6i & 9+6i & -11 \end{bmatrix}$$

Similar matrices: A square matrix A is said to be similar to a square matrix B if there exists an invertible matrix P such that $A = P^{-1}BP$. P is called similarity matrix. This relation of similarity is a symmetric relation.

Cayley Hamilton theorem: Every square matrix satisfies its own characteristic equation.

Eigen values and Eigen Vectors: Let A be a square matrix. Then the equation determinant $(A - \alpha I)$ =0 is called characteristic equation of A. The roots of characteristic equation of A are called Eigen values or latent roots of matrix A.

A column vector X satisfying the equation $AX = \alpha X$ i.e. $(A - \alpha I)X = 0$ is called Eigen vector or latent vector of matrix A corresponding to eigen value α.

Diagonalizable matrix: A square matrix A is said to be diagonalizable if there exists an invertible matrix P such that

$$P^{-1}BP = D$$

Where D is a diagonal matrix and the diagonal elements of D are Eigen values of A.

1. The characteristics equation of a matrix A is $t^2-t-1=0$, then determine A^{-1}.

Sol. By Cayley Hamilton theorem, every square matrix satisfies its characteristic equation.

Therefore A^2-A-1=0

or A^2-A=1

Premultipying both sides by A

A-I=A^{-1}

2. Prove eigen value of a Hermitian matrix is real.

Sol. Let A be a Hermitian matrix. Therefore $A^{\theta} = A ------(1)$

Let α be eigen value of A and X be the corresponding non-zero eigen vector. Then

$$AX = \alpha X \xrightarrow{yields} (AX)^{\theta} = (\alpha X)^{\theta} \xrightarrow{yields} X^{\theta}A^{\theta} = \bar{\alpha}X^{\theta} \xrightarrow{yields} X^{\theta}A = \bar{\alpha}X^{\theta} \quad \text{(using (1))}$$

Post multiplying both sides by X, we get

$$X^{\theta}(AX) = \bar{\alpha}(X^{\theta}X) \xrightarrow{yields} X^{\theta}\,\alpha X = \bar{\alpha}(X^{\theta}X) \xrightarrow{yields} \alpha(X^{\theta}X) = \bar{\alpha}(X^{\theta}X) \xrightarrow{yields} \alpha = \bar{\alpha}$$

Hence α is a real number. Therefore Eigen value of a Hermitian matrix is real.

3. Prove $\frac{|A|}{\alpha}$ is an eigen value of $adj\ (A)$eigen vector remaining the same if α is an eigen value of A and X is corresponding Eigen vector.

Sol. Let A be a square matrix$------(1)$

Let α be eigen value of A and X be the corresponding non-zero eigen vector. Then

$AX = \alpha X$ (using (1))

Pre- multiplying both sides by adj (A), we get

$$adj\ (A)(AX) = adj\ (A)\alpha X \xrightarrow{yields} (adj\ (A)A)X = \alpha(adj\ (A)X) \xrightarrow{yields} |A|X = \alpha(adj\ (A)X)$$

$$adj\ (A)X = \frac{|A|}{\alpha}X$$

Hence $\frac{|A|}{\alpha}$ is an eigen value of $adj\ (A)$ and X is corresponding Eigen vector.

4. Prove that product of two orthogonal matrices is orthogonal matrix

Sol. Let A and B be two orthogonal matrices. Therefore

$AA^T = A^TA = I$ and $BB^T = B^TB = I$

Now $(AB)(AB)^T = ABB^TA^T = AIA^T = AA^T = I$ and

$(AB)^T(AB) = B^TA^T\ AB = BIB^T = BB^T = I$

Hence AB is an orthogonal matrix. Therefore product of two orthogonal matrices is orthogonal matrix.

5. Prove that transpose of an orthogonal matrix is orthogonal matrix.

Sol. Let A be orthogonal matrix. Therefore

$AA^T = A^TA = I$

Now $A^T(A^T)^T = A^TA = I$ and

$(A^T)^TA^T = AA^T = I$

Hence A^T is an orthogonal matrix

Therefore transpose of an orthogonal matrix is orthogonal matrix.

6. Prove that inverse of an orthogonal matrix is an orthogonal matrix.

Sol. Let A be orthogonal matrix. Therefore

$AA^T = A^TA = I$

Now $A^{-1}(A^{-1})^T = A^{-1}(A^T)^{-1} = (A^TA)^{-1} = I^{-1} = I$ and

$(A^{-1})^TA^{-1} = (A^T)^{-1}A^{-1} = (AA^T)^{-1} = I^{-1} = I$

Hence A^{-1} is an orthogonal matrix

Therefore inverse of an orthogonal matrix is orthogonal matrix.

7. Prove that determinant of an orthogonal matrix is ± 1.

Sol. Let A be orthogonal matrix. Therefore

$AA^T = A^TA = I$

Taking determinant on both sides

$|AA^T| = |I| \xrightarrow{yields} |A||A^T| = 1 \xrightarrow{yields} |A||A| = 1 \xrightarrow{yields} |A|^2 = 1 \xrightarrow{yields} |A| = \pm 1$

(Because $|CD| = |C||D|, \quad |I| = 1, \quad |A| = |A^T|$)

8. Prove that inverse of a unitary matrix is an unitary matrix.

Sol. Let A be unitary matrix. Therefore

$A^\theta A = AA^\theta = I$ where $A^\theta = \left(\overline{A}\right)^T$

Now $A^{-1}(A^{-1})^\theta = A^{-1}\left(A^\theta\right)^{-1} = \left(A^\theta A\right)^{-1} = I^{-1} = I$ and

$(A^{-1})^\theta A^{-1} = \left(A^\theta\right)^{-1} A^{-1} = \left(AA^\theta\right)^{-1} = I^{-1} = I$

Hence A^{-1} is an orthogonal matrix

Therefore inverse of an orthogonal matrix is orthogonal matrix.

9. State and prove Cayley Hamilton theorem.

Sol. Statement: Every square matrix satisfies its own characteristic equation.

Proof: Let A be a square matrix of order n and its characteristic equation be $|A - \lambda I| = 0$

i.e. $(-1)^n\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots \ldots \ldots + a_n = 0$

Required to be proved: $(-1)^nA^n + a_1A^{n-1} + a_2A^{n-2} + \cdots \ldots \ldots + a_nI = 0$

Here λ is an eigen value of A.

$[A - \lambda I]$ is a matrix of order n $\xrightarrow{yields}$ $adj.(A - \lambda I)$ is a matrix of order (n-1).

Therefore we can write $adj.(A - \lambda I) = P_1\lambda^{n-1} + P_2\lambda^{n-2} + \cdots \ldots \ldots + P_n$ where $P_1, P_2, \ldots \ldots \ldots P_n$ are square matrices.

Also $A(adj.A) = |A|I \xrightarrow{yields} (A - \lambda I)adj.(A - \lambda I) = |A - \lambda I|I$

$\xrightarrow{yields} (A - \lambda I)[P_1\lambda^{n-1} + P_2\lambda^{n-2} + \cdots \ldots \ldots + P_n] = [(-1)^n\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots \ldots \ldots + a_n]I$

Comparing coefficients of like powers of A, we get

$-P_1 = (-1)^nI$

$AP_1 - P_2 = a_1I$

$AP_2 - P_3 = a_2I$

$AP_3 - P_4 = a_3I$

…………………… (and so on)

$AP_{n-1} - P_n = a_{n-1}I$

$AP_n = a_nI$

Pre-multiplying these equations by $A^n,\ A^{n-1},\ A^{n-2}, \dots \dots \dots, A,\ I$ respectively on both sides and adding, we get $0 = (-1)^n A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots \dots \dots + a_n I$

$\xrightarrow{yields} (-1)^n A^n + a_1 A^{n-1} + a_2 A^{n-2} + \cdots \dots \dots + a_n I = 0$

(Hence proved).

10. Find characteristic equation of $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

Sol. $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$

Characteristic equation of A is $|A - \alpha I| = 0 \xrightarrow{yields} \begin{vmatrix} 1-\alpha & 0 & -1 \\ 1 & 2-\alpha & 1 \\ 2 & 2 & 3-\alpha \end{vmatrix} = 0$

$\xrightarrow{yields} \alpha^3 - 6\alpha^2 + 11\alpha - 6 = 0$

11. Is $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$ diagonalizable?

Sol. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$

Characteristic equation of A is $|A - \alpha I| = 0 \xrightarrow{yields} \begin{vmatrix} 1-\alpha & 0 & 0 \\ 0 & 3-\alpha & -1 \\ 0 & -1 & 3-\alpha \end{vmatrix} = 0$

$\xrightarrow{yields} \alpha^3 - 7\alpha^2 + 14\alpha - 8 = 0 \xrightarrow{yields} \alpha = 1,2,4$

Since A has three distinct Eigen values, ∴ it has three linearly independent Eigen vectors. Hence A

A is diagonalizable.

12. Verify Cayley Hamilton theorem for $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$. Hence find A^{-1}. Also find Eigen values and vectors of A

Sol. $A = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}$

Characteristic equation of A is $|A - \alpha I| = 0 \xrightarrow{yields} \begin{vmatrix} 1-\alpha & 4 \\ 3 & 2-\alpha \end{vmatrix} = 0$

$\xrightarrow{yields} \alpha^2 - 3\alpha - 10 = 0 \xrightarrow{yields} \alpha = -2,5$

By Cayley Hamilton theorem $A^2 - 3A - 10I = 0$ ……………………….(*)

Now $A^2 = \begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 13 & 12 \\ 9 & 16 \end{bmatrix}$,

$\therefore A^2 - 3A - 10I = \begin{bmatrix} 13 & 12 \\ 9 & 16 \end{bmatrix} + \begin{bmatrix} -3 & -12 \\ -9 & -6 \end{bmatrix} + \begin{bmatrix} -10 & 0 \\ 0 & -10 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

∴ Cayley Hamilton theorem is verified for given matrix A.

Multiplying both sides of (*) by A^{-1}, we get $A - 3I = 10A^{-1} \xrightarrow{yields} A^{-1} = \frac{1}{10}\begin{bmatrix} -2 & 4 \\ 3 & -1 \end{bmatrix}$

Let $X_1 = \begin{bmatrix} x \\ y \end{bmatrix}$ be the Eigen vector of A corresponding to Eigen value $\alpha = -2$.

$$\therefore\ [A - \alpha I]X_1 = 0 \xrightarrow{yields} [A - (-2)I]X_1 = 0 \xrightarrow{yields} \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\xrightarrow{yields} 3x + 4y = 0\,, 3x + 4y = 0 \xrightarrow{yields} \frac{x}{-4} = \frac{y}{3}$

$\therefore\ X_1 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$ is the Eigen vector of A corresponding to Eigen value $\alpha = -2$.

Let $X_2 = \begin{bmatrix} x \\ y \end{bmatrix}$ be the Eigen vector of A corresponding to Eigen value $\alpha = 5$.

$$\therefore\ [A - \alpha I]X_2 = 0 \xrightarrow{yields} [A - (5)I]X_2 = 0 \xrightarrow{yields} \begin{bmatrix} -4 & 4 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\xrightarrow{yields} -4x + 4y = 0\,, 3x - 3y = 0 \xrightarrow{yields} x = y \xrightarrow{yields} \frac{x}{1} = \frac{y}{1}$

$\therefore\ X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the Eigen vector of A corresponding to Eigen value $\alpha = 5$.

13. Verify Cayley Hamilton theorem for $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$. Hence find A^{-1}.

Sol. $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Characteristic equation of A is $|A - \alpha I| = 0 \xrightarrow{yields} \begin{vmatrix} 2-\alpha & -1 & 1 \\ -1 & 2-\alpha & -1 \\ 1 & -1 & 2-\alpha \end{vmatrix} = 0$

$\xrightarrow{yields} \alpha^3 - 6\alpha^2 + 9\alpha - 4 = 0$

By Cayley Hamilton theorem $A^3 - 6A^2 + 9A - 4I = 0$ …………………..(i)

L.H.S. $A^2 = A.A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$

$$A^3 = AAA = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$$

Hence $A^3 - 6A^2 + 9A - 4I$

$$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} - 4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 22-36+18-4 & -21+30-9 & 21-30+9 \\ -21+30-9 & 22-36+18-4 & -21+30-9 \\ 21-30+9 & -21+30-9 & 22-36+18-4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence Cayley Hamilton theorem is verified for the given matrix A

From(i), $4I = A^3 - 6A^2 + 9A$

Multiplying both sides by A^{-1}, we get

$$A^{-1} = \frac{1}{4}[A^2 - 6A + 9I] = \frac{1}{4}\left[\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}\right]$$

$$= \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

14. Find Eigen values and vectors of $A = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

Sol. $A = \begin{bmatrix} 3 & 1 & -1 \\ -2 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$

Characteristic equation of A is $|A - \alpha I| = 0 \xrightarrow{yields} \begin{vmatrix} 3-\alpha & 1 & -1 \\ -2 & 1-\alpha & 2 \\ 0 & 1 & 2-\alpha \end{vmatrix} = 0$

$\xrightarrow{yields} \alpha^3 - 6\alpha^2 + 11\alpha - 6 = 0 \xrightarrow{yields} \alpha = 1,2,3$ are Eigen values of given matrix.

Let $X_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the Eigen vector of A corresponding to Eigen value $\alpha = 1$.

$$\therefore\ [A - \alpha I]X_1 = 0 \xrightarrow{yields} [A - (1)I]X_1 = 0 \xrightarrow{yields} \begin{bmatrix} 2 & 1 & -1 \\ -2 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\xrightarrow{yields} 2x + y - z = 0\,, -2x + 2z = 0, y + z = 0$

From first two equations, $\frac{x}{\begin{vmatrix} 1 & -1 \\ 0 & 2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} -1 & 2 \\ 2 & -2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 2 & 1 \\ -2 & 0 \end{vmatrix}} \xrightarrow{yields} \frac{x}{2} = \frac{y}{-2} = \frac{z}{2} \xrightarrow{yields} \frac{x}{1} = \frac{y}{-1} = \frac{z}{1}$

$\therefore\ X_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is the Eigen vector of A corresponding to Eigen value $\alpha = 1$.

Let $X_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the Eigen vector of A corresponding to Eigen value $\alpha = 2$.

$$\therefore\ [A - \alpha I]X_2 = 0 \xrightarrow{yields} [A - (2)I]X_2 = 0 \xrightarrow{yields} \begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\xrightarrow{yields} x + y - z = 0\,, -2x - y + 2z = 0, y = 0$

From first two equations, $\frac{x}{\begin{vmatrix} 1 & -1 \\ -1 & 2 \end{vmatrix}} = \frac{y}{\begin{vmatrix} -1 & 1 \\ 2 & -2 \end{vmatrix}} = \frac{z}{\begin{vmatrix} 1 & 1 \\ -2 & -1 \end{vmatrix}} \xrightarrow{yields} \frac{x}{1} = \frac{y}{0} = \frac{z}{1}$

$\therefore\ X_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is the Eigen vector of A corresponding to Eigen value $\alpha = 2$.

Let $X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the Eigen vector of A corresponding to Eigen value $\alpha = 3$.

$$\therefore\ [A - \alpha I]X_3 = 0 \xrightarrow{yields} [A - (3)I]X_3 = 0 \xrightarrow{yields} \begin{bmatrix} 0 & 1 & -1 \\ -2 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\xrightarrow{yields} y - z = 0\,, -2x - 2y + 2z = 0, y - z = 0 \xrightarrow{yields} y - z = 0\,, -2x - 2y + 2z = 0$

$\therefore$ we get , $\frac{x}{\begin{vmatrix}1 & -1\\ -2 & 2\end{vmatrix}} = \frac{y}{\begin{vmatrix}-1 & 0\\ 2 & -2\end{vmatrix}} = \frac{z}{\begin{vmatrix}0 & 1\\ -2 & -2\end{vmatrix}} \xrightarrow{yields} \frac{x}{0} = \frac{y}{2} = \frac{z}{2} \xrightarrow{yields} \frac{x}{0} = \frac{y}{1} = \frac{z}{1}$

$\therefore X_3 = \begin{bmatrix}0\\1\\1\end{bmatrix}$ is the Eigen vector of A corresponding to Eigen value $\alpha = 2$.

15. Find Eigen values and vectors of $A = \begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}$

Sol. $A = \begin{bmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}$

Characteristic equation of A is $|A - \alpha I| = 0 \xrightarrow{yields} \begin{vmatrix}1-\alpha & 1 & 0\\ 0 & 1-\alpha & 1\\ 0 & 0 & 1-\alpha\end{vmatrix} = 0$

$\xrightarrow{yields} (1-\alpha)^3 \xrightarrow{yields} \alpha = 1,1,1$ are Eigen values of given matrix.

Let $X_1 = \begin{bmatrix}x\\y\\z\end{bmatrix}$ be the Eigen vector of A corresponding to Eigen value $\alpha = 1$.

$$\therefore [A - \alpha I]X_1 = 0 \xrightarrow{yields} [A - (1)I]X_1 = 0 \xrightarrow{yields} \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\xrightarrow{yields} y = 0, z = 0.\ Take\ x = 1$

$\therefore X_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$ is the Eigen vector of A corresponding to Eigen value $\alpha = 1$.

16. Examine whether the following matrix is diagonalizable. If so, obtain the matrix P such that $P^{-1}AP$ is a diagonal matrix. $A = \begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}$

Sol. $A = \begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}$

Characteristic equation of A is $|A - \alpha I| = 0 \xrightarrow{yields} \begin{vmatrix}-2-\alpha & 2 & -3\\ 2 & 1-\alpha & -6\\ -1 & -2 & 0-\alpha\end{vmatrix} = 0$

$\xrightarrow{yields} -(\alpha+3)(\alpha+3)(\alpha-5) = 0 \xrightarrow{yields} \alpha = -3, -3, 5$ are Eigen values of given matrix.

Let $X_1 = \begin{bmatrix}x\\y\\z\end{bmatrix}$ be the Eigen vector of A corresponding to Eigen value $\alpha = -3$.

$$\therefore [A - \alpha I]X_1 = 0 \xrightarrow{yields} [A - (-3)I]X_1 = 0 \xrightarrow{yields} \begin{bmatrix}1 & 2 & -3\\ 2 & 4 & -6\\ -1 & -2 & 3\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

(Operating $R_2 \to R_2 - 2R_1, R_3 \to R_3 + R_1$)

$\xrightarrow{yields} \begin{bmatrix}1 & 2 & -3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix} \xrightarrow{yields} x + 2y - 3z = 0$

Choose $y = 0 \xrightarrow{yields} x - 3z = 0 \xrightarrow{yields} \frac{x}{3} = \frac{z}{1}$

$\therefore\ X_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ is the first Eigen vector of A corresponding to Eigen value $\alpha = -3$.

Choose $z = 0 \xrightarrow{yields} x + 2y = 0 \xrightarrow{yields} \frac{x}{-2} = \frac{y}{1}$

$\therefore\ X_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$ is another Eigen vector of A corresponding to Eigen value $\alpha = -3$.

Let $X_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be the Eigen vector of A corresponding to Eigen value $\alpha = 5$.

$$\therefore\ [A - \alpha I]X_3 = 0 \xrightarrow{yields} [A - (5)I]X_3 = 0 \xrightarrow{yields} \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\xrightarrow{yields} -7x + 2\,y - 3z = 0\,, 2x - 4y - 6z = 0, -x - 2y - 5z = 0$

$\therefore$ from first two equations we get , $\dfrac{x}{\begin{matrix} 2 & -3 \\ -4 & -6 \end{matrix}} = \dfrac{y}{\begin{matrix} -3 & -7 \\ -6 & 2 \end{matrix}} = \dfrac{z}{\begin{matrix} -7 & 2 \\ 2 & -4 \end{matrix}} \xrightarrow{yields} \frac{x}{-24} = \frac{y}{-48} = \frac{z}{24} \xrightarrow{yields} \frac{x}{1} = \frac{y}{12} = \frac{z}{-1}$

$\therefore\ X_3 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$ is the Eigen vector of A corresponding to Eigen value $\alpha = 5$.

$\therefore$ Modal Matrix P = $\begin{bmatrix} -2 & 3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$

$|P| = \begin{vmatrix} -2 & 3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{vmatrix} = 8 \neq 0$. Hence vectors are linearly independent and the given matrix is

Diagonalizable.

$$P^{-1} = \frac{Adj.P}{|P|} = \frac{1}{8}\begin{bmatrix} -2 & 4 & 6 \\ 1 & 2 & 5 \\ 1 & 2 & -3 \end{bmatrix}$$

Diagonal Matrix = D = $P^{-1}AP = \frac{1}{8}\begin{bmatrix} -2 & 4 & 6 \\ 1 & 2 & 5 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}\begin{bmatrix} -2 & 3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix}$

$$= \frac{1}{8}\begin{bmatrix} -24 & 0 & 0 \\ 0 & -24 & 0 \\ 0 & 0 & 40 \end{bmatrix}\begin{bmatrix} -2 & 3 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & -1 \end{bmatrix} = \begin{bmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

17. Let T be a linear transformation defined by $T\left[\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $T\left[\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$, $T\left[\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 \\ -2 \\ -3 \end{pmatrix}$, $T\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right] = \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$. Find $T\left[\begin{pmatrix} 4 & 5 \\ 3 & 8 \end{pmatrix}\right]$.

Sol.The matrices $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ are linearly independent and hence form a basis in the space of 2×2 matrices. We write for any scalars $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, not all zero

$$\begin{pmatrix} 4 & 5 \\ 3 & 8 \end{pmatrix} = \alpha_1\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + \alpha_2\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + \alpha_3\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} + \alpha_4\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} \alpha_1 & \alpha_1 + \alpha_2 \\ \alpha_1 + \alpha_2 + \alpha_3 & \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \end{bmatrix}$$

Comparing the elements and solving the resulting system of equations, we get

$\alpha_1 = 4, \alpha_2 = \quad 1, \alpha_3 = -2,\ \alpha_4 = 5$. Since T is a linear transformation,

$$\therefore T\left[\begin{pmatrix}4 & 5\\3 & 8\end{pmatrix}\right] = \alpha_1 T\left[\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}\right] + \alpha_2\, T\left[\begin{pmatrix}0 & 1\\1 & 1\end{pmatrix}\right] + \alpha_3\;\; T\left[\begin{pmatrix}0 & 0\\1 & 1\end{pmatrix}\right] + \alpha_4\;\; T\left[\begin{pmatrix}0 & 0\\0 & 1\end{pmatrix}\right]$$

$$= 4\begin{pmatrix}1\\2\\3\end{pmatrix} + 1\begin{pmatrix}1\\-2\\3\end{pmatrix} - 2\begin{pmatrix}1\\-2\\-3\end{pmatrix} + 5\begin{pmatrix}-1\\2\\3\end{pmatrix} = \begin{pmatrix}-2\\20\\36\end{pmatrix}$$